

# More non-locality in the three-qubit Greenberger-Horne-Zeilinger state

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The non-local properties of the noisy three-qubit Greenberger-Horne-Zeilinger (GHZ) states,  $\rho_v = v|GHZ\rangle\langle GHZ| + (1-v)\mathbb{I}/8$  parameterized by the visibility  $0 \leq v \leq 1$  are investigated. Based on the violation of the  $2 \times 2 \times 2$ -setting Mermin inequality,  $\rho_v$  is non-local for the parameter range  $1/2 < v \leq 1$ . It has been posed whether additional settings would allow to lower the threshold visibility. Here we report on Bell inequalities giving a threshold value smaller than  $v = 1/2$ . This rules out the possibility of a local hidden variable model in the limit of  $v = 1/2$ . In particular, the lowest threshold visibility we found is  $v = 0.496057$ , attainable with  $5 \times 5 \times 5$  settings, whereas the most economical one in number of settings corresponds to  $3 \times 3 \times 4$  settings. The method which enabled us to obtain these results, and in particular the about 10000 tight Bell inequalities giving  $v < 1/2$  are also discussed in detail.

## I. INTRODUCTION

Quantum theory allows correlations between remote systems, which are fundamentally different from classical correlations. Quantum entanglement is in the heart of this phenomenon [1]. In particular, separated observers may carry out local measurements on entangled quantum states on such a way that the correlations they generate are outside the set of common cause correlations. This is known as Bell's theorem [2].

Moreover, such quantum correlations may find application in novel device-independent information tasks. They enable perfect security [3, 4] or randomness generation [5] without the need to trust the internal working of the devices.

It is desirable to find states which tolerate the largest noise in order these protocols be useful. In this regard, let us note a link between the strength of violation of certain multipartite Bell inequalities and the security of quantum communication protocols [6].

For the simplest case of a Werner state [7], which is the maximally entangled singlet state mixed with white noise, the CHSH [8] seems to be the best inequality for a modest number of settings ( $\leq 10$ ) [9, 14], giving the critical visibility  $v = 1/\sqrt{2}$ . Recently, this value has been slightly overcome with 465 number of settings [10].

When we turn to a maximally entangled state shared by more than two parties, the threshold visibility drops down rapidly [11] (exponentially in the number of qubits). Even for three parties by sharing the noisy 3-qubit GHZ state the critical visibility becomes  $v_{crit} = 1/2$  using the Mermin inequality (note that the separability limit corresponds to  $v_{sep} = 1/5$  [12]). To the best of our knowledge, there has been no known Bell inequality giving a lower threshold. Let us note that in recent studies [13, 14] using a sophisticated numerical method the lowest value found up to settings  $5 \times 5 \times 5$ , matched  $v = 1/2$  associated with the Mermin inequality. In the present paper, we provide a systematic study to explore Bell inequalities with violations

beyond the Mermin value. Our findings may also bear relevance to the problem of simulating GHZ-correlations with classical communication [15, 16].

We start by introducing the Mermin inequality (Sec. II). Then, in Sec. III a heuristic method is given to generate tight Bell inequalities with potentially large violations. In Sec. IV, we present the results by listing the Bell inequalities according to different aspects. In Sec. V, a symmetric one is highlighted, with small coefficients, and a closed solution is given for the maximum quantum violation yielding a critical visibility lower than  $1/2$ . The paper ends in Sec. VI by summarizing the results and posing open questions.

## II. MERMIN INEQUALITY AS A CASE STUDY

Let us represent the  $2 \times 2 \times 2$  setting Mermin polynomial [11] in terms of three-party correlators,

$$M \equiv A_0 B_0 C_0 - A_0 B_1 C_1 - A_1 B_0 C_1 - A_1 B_1 C_0, \quad (1)$$

where the above polynomial is to be understood as a sum of expectation values; for example,  $A_1 B_1 C_0$  denotes the average value of the product of the outcomes of Alice, Bob and Cecil when both Alice and Bob performs the second measurement and Cecil performs the first one. The local bound of this inequality is  $\mathcal{L} = 2$ .

For a given number of measurement settings and outcomes the correlations which can be described by a local realistic model define a polytope (the so-called Bell polytope), which is a convex set with a finite number of extreme points. Bell inequalities define the limits on these correlations. The Mermin inequality is a facet of the  $2 \times 2 \times 2$  local 3-party-correlation polytope [17] (involving only full-tripartite correlation terms) and also for the  $2 \times 2 \times 2$  Bell polytope [18] (involving single party marginals and 2-party correlators as well). Note that in order to get the lowest critical visibility for a given number of settings and outcomes,

it is always sufficient to consider the facets of the local polytope. We refer to inequalities associated with these facets as tight Bell inequalities (hence Mermin inequality is a tight one). Furthermore, it suffices to focus on inequivalent Bell inequalities, that is, on those which are not equivalent under relabelling of measurement settings, outcomes and exchange of parties. We would also like to mention that the inequivalent Bell inequalities to be presented turn out to be tight both in the full-tripartite-correlation space and in the full probability space.

Let us now take the noisy 3-qubit GHZ state,

$$\rho_v = v|GHZ\rangle\langle GHZ| + (1-v)\frac{\mathbb{I}}{8}, \quad (2)$$

where  $|GHZ\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$  is the 3-qubit GHZ state [19], and  $v$  is the visibility parameter. With equatorial von Neumann measurements, corresponding to traceless observables, the maximum quantum value saturates the algebraic limit ( $\mathcal{Q} = 4$ ). This is the no-signalling limit as well, hence Mermin inequality exhibits pseudotelepathy [20]. Note, that the inequalities we present in the following will not have this property, that is, the no-signalling bound will never be saturated. We denote the threshold visibility for which quantum correlations turn to classical by  $v_{crit}$  (namely, for  $v > v_{crit}$  in Eq. (2) there do not exist local realistic correlations reproducing the quantum ones). From the form of Eq. (2) it follows the ratio  $v_{crit} = \mathcal{L}/\mathcal{Q}$ , where  $\mathcal{L}$  is the local bound and  $\mathcal{Q}$  is the quantum bound achievable with traceless observables and yields  $v_{crit} = 1/2$  for the 3-qubit GHZ state based on the Mermin inequality. In the following we will exhibit several inequalities which go beyond this bound.

### III. THE METHOD

To find Bell inequalities with low classical per quantum values (hence low critical visibilities), we applied a two step procedure. In this paragraph we describe the two steps briefly, and in the next ones a more detailed description is given. In the first step we used a simplex downhill algorithm [21] to minimize the classical per quantum value for Bell inequalities of very special form, characterized by just one parameter for each measurement setting. The initial parameters were chosen randomly. This procedure leads to non-tight Bell inequalities (that is, which do not define facets of the local full-tripartite correlation polytope) with not exceptionally good critical visibilities, usually between 0.5 and 0.6. In no case we got a desired value of less than 0.5 in this way. However, the best critical visibility can always be achieved with a tight Bell inequality, therefore, in the second step we determined the facet of the polytope crossed by the ray pointing towards the direction in the space of correlations defined by the Bell coefficients we

have got in the first step. The tight inequality corresponding to this facet always gives a critical visibility value better than the original inequality does. Here we worked in the restricted space of three-party correlations, but all inequalities we have got this way are tight inequalities in the full space as well, which includes two-party correlation terms and single-party marginal terms.

The coefficients of the special Bell inequalities, which multiply the corresponding three-party correlators  $A_i B_j C_k$  we considered in the first step are given as:

$$M_{ijk} = \cos(\varphi_i^A + \varphi_j^B + \varphi_k^C). \quad (3)$$

When optimizing the critical visibility we have taken  $\sum_{i=1}^{m_A} \sum_{j=1}^{m_B} \sum_{k=1}^{m_C} M_{ijk}^2$  as the quantum value of the Bell expression, that is the coordinates of the point in the space of correlations for which we calculated the quantum value are just the Bell coefficients themselves. This point does belong to the set of quantum correlations, as it can be achieved by von Neumann projective measurements performed on components of a shared 3-qubit GHZ state. If we choose the  $i$ th,  $j$ th and  $k$ th measurement operator  $\hat{A}_i$ ,  $\hat{B}_j$  and  $\hat{C}_k$  of Alice, Bob and Cecil, respectively as

$$\begin{aligned} \hat{A}_i &= \cos \varphi_i^A \hat{\sigma}_x + \sin \varphi_i^A \hat{\sigma}_y \\ \hat{B}_j &= \cos \varphi_j^B \hat{\sigma}_x + \sin \varphi_j^B \hat{\sigma}_y \\ \hat{C}_k &= \cos \varphi_k^C \hat{\sigma}_x + \sin \varphi_k^C \hat{\sigma}_y, \end{aligned} \quad (4)$$

where  $\hat{\sigma}_x$  and  $\hat{\sigma}_y$  are Pauli operators, then the joint correlation is given by the right hand side of Eq. (3), indeed, that is:

$$A_i B_j C_k = \langle \hat{A}_i \otimes \hat{B}_j \otimes \hat{C}_k \rangle = \cos(\varphi_i^A + \varphi_j^B + \varphi_k^C). \quad (5)$$

See for example Appendix C in Ref. [22] or Eqs. (6,7) in Ref. [23] for more general formulae, allowing for non-equatorial measurements (ones with  $\hat{\sigma}_z$  components).

We have chosen the form of the Bell coefficients to coincide with the coordinates of an existing point in the set of quantum correlations because for this point this is the Bell inequality achieving the largest possible quantum value divided by the length of the vector defined by the coefficients. The reason we allowed only equatorial measurements to get the point in the correlation space was that maximum quantum value for the GHZ state can always be achieved for any full-tripartite correlation-type Bell inequality without taking more general operators (see for instance [23]).

We note that the quantum value we consider as above is usually not the maximum quantum value of the Bell inequality. Even allowing only 3-qubit GHZ states there are usually quantum correlations giving larger quantum violations. However, the improvement appears usually only in the fourth or fifth significant digit by the time the optimization procedure is finished, so it is unnecessary to determine the true maximum violation. The objective function has

TABLE I: Bell inequalities with critical visibility  $v_{crit} < 0.5$ . The two cases with three measurement settings for two parties and one for one party, and the best ten cases with three measurement settings for two parties and one for the other two. Symbols  $\mathcal{L}$  and  $M_{ijk}$  denote the local bound and the Bell coefficients, respectively.

Case	$v_{crit}$	$\mathcal{L}$	$M_{ijk}$											
$V_{343}^1$	0.49967	44	4	-4	0	-10	3	4	4	-1	3	-6	14	9
			-4	-3	-4	-1	3	-3	0	-2	-3	-4	-4	1
			-6	3	14	-9	-4	-3	-4	-1	-4	4	0	10
$V_{343}^2$	0.49972	24	-2	5	3	0	4	5	-1	-8	-2	0	-2	-2
			2	5	-3	8	-2	5	3	0	2	0	2	2
			2	0	2	2	-2	0	-2	-2	-2	0	2	0
$V_{344}^1$	0.49851	16	0	2	1	-1	-1	-1	-1	1	-1	-1	1	1
			-3	-5	-1	-3	-4	2	-1	1	-1	1	0	-2
			3	-1	-2	-2	3	-3	0	0	0	2	1	-1
$V_{344}^2$	0.49860	76	-25	14	-11	-6	6	1	-7	6	13	12	4	-1
			0	14	11	7	-7	0	9	-6	15	11	3	-1
			-7	0	6	-3	3	-1	-6	-6	0	-1	-1	2
$V_{344}^3$	0.49863	24	-2	3	-2	-1	-3	1	2	0	5	1	0	-4
			2	-1	-2	1	-2	1	-2	-1	0	0	0	0
			-2	2	-2	0	-3	2	2	-1	-5	-1	0	4
$V_{344}^4$	0.49875	20	-2	3	1	-2	2	1	-1	-2	0	-4	0	-4
			2	-2	1	1	-2	-2	-1	3	0	-4	0	-4
			-2	-1	0	1	-2	1	0	-1	0	0	0	0
$V_{344}^5$	0.49876	20	0	-1	2	-1	-2	-1	-1	0	0	1	0	-1
			2	2	2	0	4	-1	1	4	-2	1	-1	0
			2	1	2	1	-2	0	2	-4	-2	2	1	1
$V_{344}^6$	0.49879	20	-2	-2	-1	-1	0	-1	1	2	2	1	0	1
			3	-2	2	1	2	1	-1	2	6	1	4	-3
			-3	2	1	-2	-2	-2	0	-2	0	-2	4	2
$V_{344}^7$	0.49881	16	0	-2	1	-1	0	-2	-1	1	3	1	0	2
			-1	-1	3	-1	0	-1	-1	2	-2	2	5	1
			1	-1	-2	0	0	1	0	1	-1	1	1	-1
$V_{344}^8$	0.49883	20	1	-1	-1	-1	-1	1	0	-2	1	-2	-1	0
			1	1	3	-1	-1	3	0	2	-2	6	-2	-2
			0	2	2	-2	0	2	2	2	3	0	-3	-2
$V_{344}^9$	0.49886	40	1	5	-3	-3	0	-4	-4	4	-8	0	-4	-4
			-1	3	2	-2	-2	0	-2	-4	0	4	2	-2
			2	-6	5	1	2	4	2	-4	-8	12	2	6
$V_{344}^{10}$	0.49891	20	-4	-4	0	0	-1	2	-1	-2	-5	0	-1	-2
			0	0	-1	1	1	-1	0	-2	2	-2	0	0
			4	4	1	-1	-2	3	1	-2	1	-6	-1	-2

to be evaluated many times during the optimization procedure, and just taking the sum of the squares of the Bell coefficients as the quantum value is much faster. In principle, allowing other than 3-qubit GHZ states, possibly states in higher dimensional Hilbert spaces might further improve the quantum value, but we have found that for all tight Bell inequalities we arrived at in the second step this was not the case. In the second step, similarly as in the work of Ref. [25], we use linear programming to find the facet of the polytope in the direction of the vector defined by the Bell coefficients we got in the first step. In this step the critical visibility improves significantly most of the case. We note that usually we got the best final results not from the initial inequalities giving the lowest critical visibilities.

Those most often led to inequalities that are equivalent to the Mermin inequality.

#### IV. RESULTS

For cases of no more than three measurement settings per party we have found no Bell inequality with critical visibility  $v_{crit}$  better than 0.5. For four measurement settings for one of the parties and three for the other two we have found two such inequivalent inequalities,  $V_{343}^1$  with  $v_{crit} = 0.49967$ , and  $V_{343}^2$  with  $v_{crit} = 0.49972$ , see Table I. We generated 30000 inequalities with the simplex downhill method starting from different random initial pa-

TABLE II: The best ten Bell inequalities with four measurement settings per party with critical visibility  $v_{crit} < 0.5$ . Symbols  $\mathcal{L}$  and  $M_{ijk}$  denote the local bound and the Bell coefficients, respectively.

Case	$v_{crit}$	$\mathcal{L}$	$M_{ijk}$																
$V_{444}^1$	0.49699	12	0	-1	1	0	-2	-3	-2	-1	0	0	0	0	2	0	-1	-1	
			0	-1	1	0	2	-1	-1	0	0	0	0	0	-2	2	0	0	
			1	1	1	1	0	-1	0	1	-1	1	0	0	0	-1	1	0	
			1	1	-1	-1	0	1	-1	0	-1	1	0	0	0	1	-2	1	
$V_{444}^2$	0.49720	12	0	0	0	0	0	-1	1	0	0	-1	1	0	0	0	0	0	
			0	1	2	1	2	0	0	2	2	-1	-1	0	0	0	-1	1	
			0	1	1	0	0	-2	1	-1	0	2	-1	1	0	1	1	0	
			0	0	-1	-1	2	-1	0	1	2	2	1	-3	0	-1	0	-1	
$V_{444}^3$	0.49751	12	1	2	-2	-1	0	0	1	-1	1	-1	1	1	0	1	0	1	
			0	-1	0	-1	0	0	0	0	0	-1	-2	1	0	2	2	0	
			0	-2	1	-1	0	-1	1	0	1	-2	0	1	1	-1	-2	0	
			1	1	1	1	0	-1	0	1	0	0	-1	1	-1	0	0	1	
$V_{444}^4$	0.49765	12	-2	-2	-1	-1	-2	-1	0	1	-1	0	1	0	-1	1	0	0	
			-1	-1	1	-1	2	0	1	1	1	-1	1	1	0	0	1	-1	
			-1	-1	1	1	1	0	-1	-2	1	0	0	1	-1	1	0	0	
			0	0	-1	1	-1	1	0	0	1	-1	0	0	0	0	1	-1	
$V_{444}^5$	0.49773	24	-4	-3	-2	-1	-3	-1	0	4	-2	3	-1	0	-1	-1	1	-1	
			-3	0	-1	2	-1	3	4	0	3	-2	0	1	-1	1	-1	-1	
			-2	-1	2	1	0	4	1	-1	-1	0	2	-3	1	-1	1	1	
			-1	2	1	2	4	0	-1	-1	0	1	-3	2	-1	-1	1	-1	
$V_{444}^6$	0.49785	16	-3	-3	-2	-2	-2	-1	-1	2	-2	2	0	0	-1	0	1	0	
			-2	1	-1	0	-1	2	1	0	1	-2	2	-1	0	-1	0	1	
			-1	-1	0	2	0	2	0	-2	-2	1	1	0	1	0	-1	0	
			0	-1	1	0	1	-1	0	0	-1	-1	-1	-1	0	-1	0	1	
$V_{444}^7$	0.49786	24	-3	-3	-2	-2	-2	3	-2	-1	-1	-2	0	3	0	-2	0	-2	
			-3	-1	-2	2	3	0	1	2	-2	1	3	0	-2	2	0	0	
			-2	-2	1	1	-2	1	0	-3	0	3	1	-2	0	0	0	0	
			-2	2	1	3	-1	2	-3	2	3	0	-2	-1	-2	0	0	-2	
$V_{444}^8$	0.49795	24	-1	2	1	-2	2	-2	0	2	-1	0	-1	0	2	2	0	0	
			-2	0	0	-6	1	2	-1	-2	-2	-4	0	-2	-1	2	1	-2	
			-1	-4	0	-3	1	-1	1	-1	-1	3	0	4	-1	0	-1	0	
			-2	-2	1	3	-2	3	0	1	-2	1	-1	2	-2	2	0	-2	
$V_{444}^9$	0.49803	24	-2	-4	-2	0	3	2	-1	2	-1	2	2	-1	0	0	-1	1	
			0	-2	-1	-1	-1	1	0	0	-2	0	-1	-1	1	-1	0	0	
			2	-6	-1	1	-2	1	-4	-1	2	-2	-2	-2	2	1	1	-2	
			0	4	-4	0	0	4	-3	-1	-3	0	1	0	1	0	0	-3	
$V_{444}^{10}$	0.49806	20	0	2	-1	-1	-1	-1	0	0	0	0	1	1	2	1	-2	-3	
			0	-2	-1	-1	-1	-1	0	0	0	0	1	-3	-2	1	-2	2	1
			0	0	2	2	0	0	2	-2	0	-2	-4	-2	0	2	0	-6	
			0	0	0	0	0	0	2	-2	0	0	-2	2	0	0	-4	4	

parameter values. In the second step we got inequalities equivalent with  $V_{343}^1$  374 times, and with  $V_{343}^2$  68 times, respectively. The rest has  $v_{crit} = 0.5$ , many of them equivalent to the Mermin inequality, or  $v_{crit} > 0.5$ . We have done the same number of attempts for three measurement settings for one of the parties and four for the other two. We arrived 1372 times at inequalities with  $v_{crit} < 0.5$ , among them there were 70 independent ones, one of them equivalent with  $V_{343}^2$ . Many of the inequalities were found only once, which indicates that there are certainly a lot we missed. The ten best ones we have got are shown in Table I.

For four measurement settings per party we have made 10000 attempts. In about one third of these attempts we got inequalities with  $v_{crit} < 0.5$ , of which over one thousand were inequivalent. The ten with the best critical visibilities are shown in Table II. It is interesting, that the integer-valued Bell coefficients are quite small for all of them. In Table II we show five more inequalities with four measurement settings per party with special properties. Two of them have only Bell coefficients 0 and  $\pm 1$ , while the other three inequalities have coefficients symmetric in the exchange of all three parties.

TABLE III: Some special Bell inequalities with four measurement settings per party with critical visibility  $v_{crit} < 0.5$ . For two cases the absolute value of the Bell coefficients are no more than one, while three cases are symmetric for all participants. Symbols  $\mathcal{L}$  and  $M_{ijk}$  denote the local bound and the Bell coefficients, respectively.

Case	$v_{crit}$	$\mathcal{L}$	$M$															
$V_{444}^{U1}$	0.49890	8	-1	-1	-1	-1	-1	-1	-1	1	-1	0	0	-1	-1	0	0	1
			-1	0	0	-1	1	-1	0	0	0	1	0	1	0	0	0	0
			0	-1	-1	0	0	1	0	1	-1	0	1	0	1	0	0	-1
			0	0	0	0	0	-1	1	0	0	-1	1	0	0	0	0	0
$V_{444}^{U2}$	0.49955	8	-1	-1	-1	-1	-1	0	0	1	-1	0	1	0	-1	1	0	0
			-1	0	0	1	0	0	1	1	1	0	1	0	0	0	0	0
			0	-1	0	1	1	0	1	0	0	0	-1	-1	-1	1	0	0
			0	0	-1	1	0	0	0	0	0	0	-1	1	0	0	0	0
$V_{444}^{S1}$	0.49895	128	-22	-10	-3	-1	-10	4	-1	-13	-3	-1	11	-1	-1	-13	-1	9
			-10	4	-1	-13	4	10	13	1	-1	13	-12	-4	-13	1	-4	12
			-3	-1	11	-1	-1	13	-12	-4	11	-12	-9	-12	-1	-4	-12	-15
			-1	-13	-1	9	-13	1	-4	12	-1	-4	-12	-15	9	12	-15	16
$V_{444}^{S2}$	0.49903	12	-3	-1	0	0	-1	0	-1	0	0	-1	1	0	0	0	0	2
			-1	0	-1	0	0	1	0	-1	-1	0	1	0	0	-1	0	-1
			0	-1	1	0	-1	0	1	0	1	1	1	1	0	0	1	-1
			0	0	0	2	0	-1	0	-1	0	0	1	-1	2	-1	-1	0
$V_{444}^{S3}$	0.49990	400	-62	-53	-43	-30	-53	34	-21	-8	-43	-21	-1	35	-30	-8	35	-7
			-53	34	-21	-8	34	-15	19	-26	-21	19	44	-16	-8	-26	-16	-12
			-43	-21	-1	35	-21	19	44	-16	-1	44	-11	-6	35	-16	-6	17
			-30	-8	35	-7	-8	-26	-16	-12	35	-16	-6	17	-7	-12	17	44

We have also made 10000 calculations for five measurement settings per party. In almost three quarters of the trials we got result with  $v_{crit} < 0.5$ . The best inequality, and some more inequalities have been found several times, but we have arrived at the majority of them only once. Some examples are shown in Tables IV and V. The improvement in  $v_{crit}$  compared to the four measurement settings per party case is not very impressive. We show the three best cases, the second best in Table V separately, for typographic reason: the integer Bell coefficients are fairly large for that case. In Table IV we also show some special cases.  $V_{555}^{S1}$  is symmetric in all participants (the analytical solution of this inequality will be given in Sec. V), while  $V_{555}^{U1}$  and  $V_{555}^{U2}$  have only coefficients 0 and  $\pm 1$ . In a few cases we got inequalities equivalent to ones with less than five measurement settings per party. One example is  $V_{455}^1$ , with only four measurement settings for Alice, for which  $v_{crit}$  is only very marginally worse than for  $V_{555}^1$ .  $V_{454}^{U1}$  has only 0 and  $\pm 1$  Bell coefficients and two participants have 4 measurement settings, only one has five. The total number of measurement settings is also 13 for  $V_{553}^{U1}$ , which also has Bell coefficients of absolute values at most one, and which is symmetric in the coefficients of Alice and Bob, who have five measurement settings each, while Cecil has only three. For this case  $v_{crit}$  is quite small as well.

For the critical visibility, we calculated the quantum bound with von Neumann projective measurements performed on components of a shared 3-qubit GHZ state with a see-saw method, similar to Ref. [24]. For all cases such

calculation has given the exact quantum bound. This is true not only for the cases we reported here, but for all tight three-party full-correlation type Bell inequalities (nonzero coefficients only for three party correlators) we checked, including the ones with worse critical visibilities. We checked all cases we got with no more than four measurement settings per party, and also several larger cases, including ones with five measurement settings per party. We have done that by calculating the upper bound for the quantum value using semidefinite programming with the method invented by Navascués, Pironio and Acín (NPA) [26]. For the calculation we used the code CSDP of Brian Borchers [27]. In all cases we found the upper bound coincides with the result of direct calculation, which is actually a lower bound. In applying the NPA method, it was enough to do it at level one  $+ab + ac + bc + abc$  in all cases. The notion of levels, and their notation is explained in Ref. [28], and also in Ref [29].

## V. STUDY OF A BELL INEQUALITY PROVIDING VISIBILITY LESS THAN 1/2.

We consider a Bell scenario where each of the three parties can choose between five possible binary measurements. We denote the outcome of measurement  $j = 0, 1, \dots, 4$  for Alice by  $A_j$  ( $B_j$  and  $C_j$  for Bob and Cecil respectively). Here we shall focus on the only symmetric inequality we found for this scenario (and can be found in Table IV),

TABLE IV: Examples of Bell inequalities with critical visibility  $v_{crit} < 0.5$  with five measurement settings for at least one party. Symbols  $\mathcal{L}$  and  $M_{ijk}$  denote the local bound and the Bell coefficients, respectively.

Case	$v_{crit}$	$\mathcal{L}$	$M_{ijk}$																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																									
$V_{555}^1$	0.496057	24	2	-2	1	-1	2	0	2	0	-2	0	0	-2	0	0	2	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

which is given by the expression:

$$\begin{aligned}
V_{555}^{S1} = & \text{sym}[-A_0B_0C_2 - A_0B_0C_3 + A_0B_2C_4 \\
& + A_0B_3C_4 + A_1B_1C_2 - A_1B_1C_3 - A_1B_2C_2 \\
& + A_1B_3C_3 - 2A_2B_2C_2 + A_2B_2C_3 - A_2B_3C_3 \\
& + A_2B_4C_4 + 2A_3B_3C_3 + A_3B_4C_4], \quad (6)
\end{aligned}$$

where the designation  $\text{sym}[X]$  indicates that the expression  $X$  is symmetrized over all the parties, for instance  $\text{sym}[A_0B_0C_2] = A_0B_0C_2 + A_0B_2C_0 + A_2B_0C_0$ . In this  $5 \times 5 \times 5$  Bell inequality the local bound is  $\mathcal{L} = 12$  (all local probability distributions satisfy the inequality  $V_{555}^{S1} \leq 12$ ).

We now show that by performing local measurements on

a 3-qubit GHZ state,

$$|GHZ\rangle = (|000\rangle + |111\rangle)/\sqrt{2}, \quad (7)$$

it is possible to obtain the value of  $\mathcal{Q} = 24.1699$  for the Bell expression (6) above. This implies the visibility of  $v_{crit} = 12/24.1699 = 0.496485$  showed in Table IV.

The local measurements used by each party are of the simple form  $\hat{A}_j = \cos \varphi_j \hat{\sigma}_x + \sin \varphi_j \hat{\sigma}_y$  (with  $j = 0, 1, \dots, 4$ ), where  $\hat{\sigma}_x$  and  $\hat{\sigma}_y$  are Pauli matrices. Moreover, the five local measurements are the same for each party, i.e.  $\hat{A}_j = \hat{B}_j = \hat{C}_j$ ,  $j = 0, 1, 2, 3, 4$ . In this case the 3-party



TABLE V: The Bell coefficients of  $V_{555}^2$ , the Bell inequality with five measurement settings per party with the critical visibility  $v_{crit} = 0.496059$ , the second best one we found. The local bound is  $\mathcal{L} = 13180$ .

-271 -124 -418 522 251	922 355 8 522 -37	540 206 69 90 -641	-294 437 853 90 -212	65 710 210 0 -565
-514 251 736 596 -403	-138 -45 -249 -612 736	-217 -641 -350 0 932	853 -212 -45 82 -514	418 -557 -400 -98 -1277
-446 271 -1146 -457 514	-636 -922 -482 612 130	546 -540 -996 -167 209	-347 294 -477 175 -853	775 -65 -145 147 -418
-457 -522 661 0 -596	604 -514 858 -336 612	-159 -90 -495 426 0	159 -90 763 604 -90	147 -8 -465 -514 90
1146 -418 669 -661 736	466 8 -1915 -858 -233	996 69 368 503 -342	461 853 342 -771 -37	145 210 -510 465 -400

correlation term has the following form (see Eq. (5)),

$$A_i B_j C_k = \cos(\varphi_i + \varphi_j + \varphi_k). \quad (8)$$

The optimal measurements are given by  $\varphi_0 = 5\pi/8$ ,  $\varphi_1 = \pi/2$ ,  $\varphi_4 = \pi/8$ , and  $\varphi_2 + \varphi_3 = 3\pi$ . Leaving  $\varphi_2$  as a free parameter, we obtain the following closed formula for the Bell value,

$$S = -12 \cos 3\varphi_2 + 12 \cos \varphi_2 (\sin \varphi_2 - 1) - 12\sqrt{2} \sin \varphi_2. \quad (9)$$

Our task is now to perform optimization over the angle  $\varphi_2$ . By inserting  $\varphi_2 = 3.73842$  into (9), we obtain the value of  $Q = 24.1699$  for the expression (9).

## VI. CONCLUSION

The non-local properties of the noisy three-qubit Greenberger-Horne-Zelinger (GHZ) states (defined by Eq. (2)), parameterized by the visibility  $0 \leq v \leq 1$  were investigated. Based on the violation of the  $2 \times 2 \times 2$ -setting Mermin inequality,  $\rho_v$  is non-local for the parameter range  $1/2 < v \leq 1$ . In this study we presented several Bell inequalities which beat the limit of  $v = 1/2$ , thereby answering the question raised by Kaszlikowski et al. [13]. In particular, the lowest threshold visibility we found is  $v = 0.496057$ , attainable with  $5 \times 5 \times 5$  settings, whereas the most economical one overcoming  $v = 1/2$  corresponds to  $3 \times 3 \times 4$  settings. The method which enabled us to obtain these results, and in particular the about 10000 tight Bell inequalities going below  $v = 1/2$  were also discussed in detail.

We expect that the presented method in Sec. III would be efficient to test the non-locality of other states as well. For

instance, in the two-party 2-qubit scenario it might prove useful to find Werner states with better visibility thresholds than the presently known ones [10] (or to find ones which overcome the  $1/\sqrt{2}$  limit corresponding to the CHSH inequality with fewer number of settings). Our technique may also be suitable to generate Bell inequalities for 3-qubit W states [30] or 4-qubit Dicke states [31], cluster states [32] with improved noise resistance.

Further, it would be interesting to study whether the technique presented could be applied to find Bell violation with lower detection efficiencies than the presently known ones both for bipartite and multipartite settings. In the bipartite case, according to Ref. [33], one may postulate the  $d \times d$  state to be  $|\psi\rangle = \epsilon|00\rangle + |11\rangle + \dots + |d-1, d-1\rangle$  (here the state is written in an unnormalized form),  $\epsilon$  is a small parameter and the measurements are rank-1 projectors. Then, by using appropriate parametrization, one might carry out the heuristic search of Sec. III for 2-outcome Bell inequalities to lower the detection efficiency required for closing the detection loophole.

In the tripartite case, it is known that for the GHZ state the minimum detection efficiency based on the Mermin inequality is 75% [34]. On the other hand, the best protocol which simulates GHZ correlations (arising from equatorial von Neumann measurements) with detection efficiencies of 50% is due to the recent work of Ref. [16]. It would be interesting to decrease the gap by considering inequalities with more than 2 settings [35] in conjunction with the technique used in this paper.

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[1] R. Horodecki et al., Rev. Mod. Phys. **81**, 865 (2009).

[2] J.S. Bell, Physics **1** (1964), 195.

[3] A.K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).

[4] A. Acín, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Phys. Rev. Lett. **98**, 230501 (2007).

[5] S. Pironio et al., Nature **464**, 1021 (2010).

[6] V. Scarani and N. Gisin, Phys. Rev. Lett. **87**, 117901 (2001).

[7] R.F. Werner, Phys. Rev. A **40**, 4277 (1989).

[8] J.F. Clauser, M.A. Horne, A. Shimony and R.A. Holt, Phys. Rev. Lett. **23**, 880 (1969).

[9] M. Żukowski, D. Kaszlikowski, A. Baturó, and J.-A. Larsson, arxiv:quant-ph/9910058 (1999).

- [10] T. Vértesi, Phys. Rev. A **78**, 032112 (2008).
- [11] N.D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990); M. Ardehali, Phys. Rev. A **46**, 5375 (1992); A.V. Belinskii and D.N. Klyshko, Phys. Usp. **36**, 653 (1993).
- [12] W. Dür, J.I. Cirac, and R. Tarrach, Phys. Rev. Lett. **83**, 3562 (1999).
- [13] D. Kaszlikowski and M. Żukowski, Int. J. Theor. Phys. **42**, 1023 (2003); arXiv:quant-ph/0302165 (2003).
- [14] J. Gruca, W. Laskowski, M. Żukowski, N. Kiesel, W. Wieczorek, C. Schmid, H. Weinfurter, Phys. Rev. A **82**, 012118 (2010).
- [15] A. Broadbent, P.-R. Chouha, and A. Tapp, Third International Conference on Quantum, Nano, and Micro Technologies pp. 5962 (2009); J.-D. Bancal, C. Branciard, and N. Gisin, Adv. Math. Phys. Article ID 293245 (2010); C. Palazuelos, D. Perez-Garcia, and I. Villanueva, arXiv:1006.5318 (2010);
- [16] C. Branciard, N. Gisin, Phys. Rev. Lett. **107**, 020401 (2011).
- [17] R.F. Werner and M. Wolf, Phys. Rev. A **64**, 032112 (2001).
- [18] C. Śliwa, Phys. Lett. A **317**, 165 (2003).
- [19] D.M. Greenberger, M.A. Horne, and A. Zeilinger, *Bells Theorem, Quantum Theory, and Conceptions of the Universe* (ed. M. Kafatos, Kluwer Academic, Dordrecht, Holland, 1989), pp. 69-72.
- [20] G. Brassard, A. Broadbent, A. Tapp, Foundations of Physics **35**, 1877 (2005).
- [21] J.A. Nelder and R. Mead, Comput. J. **7**, 308 (1965).
- [22] J.-D. Bancal, N. Gisin, Y.-C. Liang, and S. Pironio, Phys. Rev. Lett. **106**, 250404 (2011).
- [23] K.F. Pál and T. Vértesi, Phys. Rev. A **83**, 062123 (2011).
- [24] K.F. Pál and T. Vértesi, Phys. Rev. A **82**, 022116 (2010).
- [25] S. Massar, S. Pironio, J. Roland, and B. Gisin, Phys. Rev. A **66**, 052112 (2002).
- [26] M. Navascués, S. Pironio, and A. Acín, Phys. Rev. Lett. **98**, 010401 (2007).
- [27] B. Borchers, Optimization Methods and Software **11**, 597, (1999); B. Borchers, Optimization Methods and Software **11**, 613, (1999); B. Borchers and J.G. Young, Computational Optimization and Applications **37**, 355 (2007); <https://projects.coin-or.org/Csdp/>.
- [28] M. Navascués, S. Pironio, and A. Acín, New. J. Phys. **10**, 073013 (2008).
- [29] K.F. Pál and T. Vértesi, Phys. Rev. A **79**, 022120 (2009).
- [30] W. Dür, G. Vidal, and J.I. Cirac, Phys. Rev. A **62**, 062314 (2000).
- [31] R.H. Dicke, Phys. Rev. A **93**, 99 (1954).
- [32] H.J. Briegel and R. Raussendorf, Phys. Rev. Lett. **86**, 910 (2001).
- [33] T. Vértesi, S. Pironio, and N. Brunner, Phys. Rev. Lett. **104**, 060401 (2010).
- [34] J.-A. Larsson, Phys. Rev. A **57**, R3145 (1998); *ibid* **59**, 4801 (1999). A. Cabello, D. Rodriguez, I. Villanueva, Phys. Rev. Lett. **101** 120402 (2008).
- [35] S. Pironio, private communication.